Self-similar branching Markov chains

Nathalie Krell

Self-similar branching Markov chains

Nathalie Krell

January 24, 2008

Laboratoire de Probabilités et Modèles Aléatoires,
Université Pierre et Marie Curie,
175 rue du Chevaleret, 75013 Paris, France.

Abstract

The main purpose of this work is to study self-similar branching Markov chains. First we will construct such a process. Then we will establish certain Limit Theorems using the theory of self-similar Markov processes.

Key Words. Branching process, Self-similar Markov process, Tree of generations, Limit Theorems.

A.M.S. Classification. 60J80, 60G18, 60F25, 60J27.

e-mail. krell@ccr.jussieu.fr

1 Introduction.

This work is a contribution to the study of a special type of branching Markov chains. We will construct a continuous time branching chain $\mathbf{X}$ which has a self-similar property and which takes its values in the space of finite point measures of $\mathbb{R}_+^*$. This type of process is a generalization of a self-similar fragmentation (see [4]), which may apply to cases where the size models non additive quantities as e.g. surface energy in aerosols. We will focus on the case where the index of self-similarity α is non-negative, which means that the bigger individuals will reproduce faster than the smaller ones. There is no loss of generality by considering this model, as the map $x \to x^{-1}$ on atoms in $\mathbb{R}_+^*$ transforms a self-similar process with index α into another one with index $-\alpha$ (and preserves the Markov property).

In this article we choose to construct the process by bare hand. We extend the method used in [4] to deal with more general processes where we allow an individual to have a mass bigger than that of its parent. We will explain in the sequel, which difficulties this new set-up entails. There exists closely related articles about branching processes, like among others [18], [19] from Kyprianou and [12], [13] from Chauvin. However notice that the time of splitting of the process depends on the size of the atoms of the process.

More precisely we will first introduce a branching Markov chains as a marked tree and we will obtain a process, indexed by generations (it is simply a random mark on the tree of generation, see Section 2). Thanks to a martingale which is associated to the latter and the theory of random stopping lines on a tree of generation, we will define the process indexed by time. After having constructed the process, we will study the evolution of the randomly chosen branch of the chain, from which we shall deduce some Limit Theorems, relying on the theory of self-similar Markov processes. In an appendix we will consider the intrinsic process and give some properties in the spirit of the article of Jagers [15]. By the way we will show properties about the earlier martingale.

2 The marked tree.

In this part we will introduce a branching Markov chain as a marked tree, which gives a genealogic description of the process that we will construct. This terminology comes from Neveu in [21] even if here the marked tree we consider is slightly different. First we introduce some notations and definitions.

A finite point measure on $\mathbb{R}_+^*$ is a finite sum of Dirac point masses $\mathbf{s} = \sum_{i=1}^n \delta_{s_i}$, where the s_i are called the atoms of $\mathbf{s}$ and $n \geq 0$ is an arbitrary integer. We shall often write $\sharp\mathbf{s} = n = \mathbf{s}(\mathbb{R}_+^*)$ for the number of atoms of $\mathbf{s}$, and $\mathcal{M}_p(\mathbb{R}_+^*)$ for the space of finite point measures on $\mathbb{R}_+^*$. We also define for $f : \mathbb{R}_+^* \to \mathbb{R}$ measurable function and $\mathbf{s} \in \mathcal{M}_p(\mathbb{R}_+^*)$

$$\langle f, \mathbf{s} \rangle := \sum_{i=1}^{\sharp s} f(s_i),$$

by taking the sum over the atoms of $\mathbf{s}$ repeated according to their multiplicity and we will sometimes use the slight abuse of notation

$$\langle f(x), \mathbf{s} \rangle := \sum_{i=1}^{\sharp s} f(s_i)$$

when f is defined as a function depending on the variable x. We endow the space $\mathcal{M}_p(\mathbb{R}_+^*)$ with the topology of weak convergence, which means that $\mathbf{s}_n$ converge to $\mathbf{s}$ if and only if $\langle f, \mathbf{s}_n \rangle$ converge to $\langle f, \mathbf{s} \rangle$ for all continuous bounded functions f.

Let $\alpha \geq 0$ be an index of self-similarity and ν be some **probability measure** on $\mathcal{M}_p(\mathbb{R}_+^*)$. The aim of this work is to construct a branching Markov chain $\mathbf{X} = ((\sum_{i=1}^{\sharp\mathbf{X}(t)} \delta_{X_i(t)})_{t\geq 0})$ with values in $\mathcal{M}_p(\mathbb{R}_+^*)$, which is self-similar with index α and has reproduction law ν. The index of self-similarity will play a part in the rate at which an individual will reproduce and the reproduction law ν will specify the distribution of the offspring. We stress that our setting includes the case when

$$\nu(\exists i \ : \ s_i > 1) > 0, \tag{1}$$

which means that with a positive probability the size of a daughter can exceed that of her mother.

To do that, exactly as described in Chapter 1 section 1.2.1 of [4], we will construct a marked tree.

We consider the Ulam Harris labelling system

$$\mathcal{U} := \cup_{n=0}^{\infty} \mathbb{N}^n,$$

with the notation $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}^0 = \{\emptyset\}$. In the sequel the elements of $\mathcal{U}$ are called nodes (or sometimes also individuals) and the distinguished node $\emptyset$ the root. For each $u = (u_1, ..., u_n) \in \mathcal{U}$, we call n the *generation* of u and write $|u| = n$, with the obvious convention $|\emptyset| = 0$. When $n \geq 0$, $u = (u_1, ..., u_n) \in \mathbb{N}^n$ and $i \in \mathbb{N}$, we write $ui = (u_1, ..., u_n, i) \in \mathbb{N}^{n+1}$ for the i-th child of u. We also define for $u = (u_1, ..., u_n)$ with $n \geq 2$,

$$mu = (u_1, ..., u_{n-1})$$

the *mother* of u, $mu = \emptyset$ if $u \in \mathbb{N}$. If $v = m^n u$ for some $n \geq 0$ we write $v \preceq u$ and say that u *stems* from v. Additionally for M a set of $\mathcal{U}$, $M \preceq v$ means that $u \preceq v$ for some $u \in M$. Generally we write $M \preceq L$ if all $x \in L$ stem from M.

Here it will be convenient to identify the point measure $\mathbf{s}$ with the infinite sequence $(s_1, ..., s_n, 0, ...)$ obtained by aggregation of infinitely many 0's to the finite sequence of the atoms of $\mathbf{s}$.

In particular we say that a random infinite sequence $(\xi_i, i \in \mathbb{N})$ has the law ν, if there is a (random) index n such that $\xi_i = 0 \Leftrightarrow i > n$ and the finite point measure $\sum_{i=1}^{n} \delta_{\xi_i}$ has the law ν.

Definition 1. *Let two independent families of i.i.d. variables be indexed by the nodes of the tree, $(\overline{\xi}_u, u \in \mathcal{U})$ and $(\mathbf{e}_u, u \in \mathcal{U})$, where for each $u \in \mathcal{U}$ $\overline{\xi}_u = (\widetilde{\xi}_{ui})_{i \in \mathbb{N}}$ is distributed according to the law ν, and $(\mathbf{e}_{ui})_{i \in \mathbb{N}}$ is a sequence of i.i.d. exponential variables with parameter 1. We define recursively for some fixed $x > 0$*

$$\xi_\emptyset := x, \qquad a_\emptyset := 0, \qquad \zeta_\emptyset := x^{-\alpha}\mathbf{e}_\emptyset,$$

and for $u \in \mathcal{U}$ and $i \in \mathbb{N}$:

$$\xi_{ui} := \widetilde{\xi}_{ui}\xi_u, \qquad a_{ui} := a_u + \zeta_u, \qquad \zeta_{ui} := \xi_{ui}^{-\alpha}\mathbf{e}_{ui}.$$

To each node u of the tree $\mathcal{U}$, we associate the mark (ξ_u, a_u, ζ_u) where ξ_u is the size, a_u the birth-time and ζ_u the lifetime of the individual with label u. We call

$$T_x = ((\xi_u, a_u, \zeta_u)_{u \in \mathcal{U}})$$

a marked tree with root of size x, and the law associated is denoted by $\mathbb{P}_x$. Let $\bar{\Omega}$ be the set of all the possible marked trees.

The size of the individuals $(\xi_u, u \in \mathcal{U})$ defines a multiplicative cascade (see the references in Section 3 of [5]). However the latter is not sufficient to construct the process $\mathbf{X}$, in fact we also need the information given by $((a_u, \zeta_u), u \in \mathcal{U})$.

Another useful concept is that of *line*. A subset $L \subset \mathcal{U}$ is a line if for every $u, v \in L$, $u \preceq v \Rightarrow u = v$. The *pre-L-sigma algebra* is

$$\mathcal{H}_L := \sigma(\widetilde{\xi_u}, \mathbf{e}_u; \exists l \in L : u \preceq l).$$

A random set of individuals

$$\mathcal{J} : \bar{\Omega} \to \mathcal{P}(\mathcal{U})$$

is *optional* if $\{\mathcal{J} \preceq L\} \in \mathcal{H}_L$ for all line $L \subset \mathcal{U}$, where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$. An *optional line* is a random line which is optional. For any optional set $\mathcal{J}$ we define the pre-$\mathcal{J}$-algebra by:

$$A \in \mathcal{H}_{\mathcal{J}} \Leftrightarrow \forall L \ \text{ line } \ \subset \mathcal{U} : A \cap \{\mathcal{J} \preceq L\} \in \mathcal{H}_L.$$

The first result is:

Lemma 1. *The marked tree constructed in Definition 1 satisfies the strong Markov branching property: for $\mathcal{J}$ an optional line and $\varphi_u : \bar{\Omega} \to [0,1]$, $u \in \mathcal{U}$, measurable functions, we get that,*

$$\mathbb{E}_1 \left(\prod_{u \in \mathcal{J}} \varphi_u \circ T^{\xi_u} \middle| \mathcal{H}_{\mathcal{J}} \right) = \prod_{u \in \mathcal{J}} \mathbb{E}_{\xi_u}(\varphi_u),$$

where T^{ξ_u} is the marked tree extracted from T_1 at the node (ξ_u, a_u, ζ_u). More precisely

$$T^{\xi_u} = ((\xi_{uv}, a_{uv} - a_u, \zeta_{uv})_{v \in \mathcal{U}}).$$

Proof. Thanks to the i.i.d properties of the random variables $(\widetilde{\xi_u}, u \in \mathcal{U})$ and $(\mathbf{e}_u, u \in \mathcal{U})$, the Markov property for lines is of course easily checked. In order to get the result for a more general optional line, we use Theorem 4.14 of [15]. Indeed, the tree we have constructed is a special case of the tree constructed by Jagers in [15]. In our case the Jagers's notation ρ_u, τ_u and σ_u are such that the type ρ_u of $u \in \mathcal{U}$, is the mass of u: ξ_u, the birth time σ_u is a_u and τ_u is here equal to ζ_{mu} (because the mother dies when she gives birth to her daughters). We notice that all the sisters have the same birth time, which means that for all $u \in \mathcal{U}$ and all $i \in \mathbb{N}$, we have that τ_{ui} is here equal to ζ_u. $\square$

3 Malthusian hypotheses and the intrinsic martingale.

We introduce some notations to formulate the fundamental assumptions of this work:

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, \mathbf{s} \rangle \nu(d\mathbf{s}) < \infty \right\},$$

and

$$p_\infty := \inf \left\{ p > \underline{p} : \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, \mathbf{s} \rangle \nu(d\mathbf{s}) = \infty \right\}$$

(with the convention $\inf \emptyset = \infty$) and then for every $p \in (\underline{p}, p_\infty)$:

$$\kappa(p) := \int_{\mathcal{M}_p(\mathbb{R}_+^*)} (1 - \langle x^p, \mathbf{s} \rangle)\, \nu(d\mathbf{s}).$$

Note that κ is a continuous and concave function (but not necessarily a strictly increasing function) on $(\underline{p}, p_\infty)$, as $p \to \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^p, \mathbf{s} \rangle \nu(d\mathbf{s})$ is a convex application. By concavity, the equation $\kappa(p) = 0$ has at most two solutions on $(\underline{p}, p_\infty)$. When a solution exists, we denote by $p_0 := \inf\{p \in (\underline{p}, p_\infty) : \kappa(p) = 0\}$ the smallest, and call p_0 the Malthusian exponent.

We now make the fundamental:

Malthusian Hypotheses. *We suppose that the Malthusian exponent p_0 exists, that $p_0 > 0$, and that*

$$\kappa(p) > 0 \quad \textit{for some} \quad p > p_0. \tag{2}$$

Furthermore we suppose that the integral

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} (\langle x^{p_0}, \mathbf{s} \rangle)^p\, \nu(d\mathbf{s}) \tag{3}$$

is finite for some $p > 1$.

Throughout the rest of this article, these hypotheses will always be taken for granted.

Note that (2) always holds when $\nu(s_i \le 1 \text{ for all } i) = 1$ (fragmentation case). We stress that κ may not be strictly increasing, and may not be negative when p is sufficiently large (see Subsection 6.1 for a consequence of this fact.)

We will give one example based on the Dirichlet process (see the book Kingman [16]). Fix $n \ge 2$, $(\upsilon_1, ..., \upsilon_n)$ n positive real numbers and $\upsilon = \sum_{i=1}^n \upsilon_i$. We define the simplex Δ_n by

$$\Delta_n := \left\{ (p_1, p_2, ..., p_n) \in \mathbb{R}_+^n,\ \sum_{j=1}^n p_i = 1 \right\}.$$

The Dirichlet distribution of parameter $(\upsilon_1, ..., \upsilon_n)$ over the simplex Δ_n has the density (with respect to the $(n-1)$-dimensional Lebesgue measure on Δ_n):

$$f(p_1, ..., p_n) = \frac{\Gamma(\upsilon)}{\Gamma(\upsilon_1)...\Gamma(\upsilon_n)} p_1^{\upsilon_1 - 1} ... p_n^{\upsilon_n - 1}.$$

Let $a := \upsilon(\upsilon+1)/(\sum_{i=1}^n \upsilon_i(\upsilon_i+1))$. Note that a is strictly larger than 1. Let the reproduction measure be the law of $(aX_1, ..., aX_n)$, where $(X_1, ..., X_n)$ is a random vector with Dirichlet distribution of parameter $(\upsilon_1, ..., \upsilon_n)$. Therefore

$$\kappa(p) = a^p \frac{\Gamma(\upsilon)}{\Gamma(\upsilon+p)} \sum_{i=1}^n \frac{\Gamma(p+\upsilon_i)}{\Gamma(\upsilon_i)},$$

$\underline{p} = -\upsilon$, $p_0 = 1$ and the Malthusian hypotheses are verified.

In this article we will call *extinction* the event that for some $n \in \mathbb{N}$, all nodes u at the n-th generation have zero size, and *non-extinction* the complementary event. We see that the probability of extinction is always strictly positive whenever $\nu(s_1 = 0) > 0$, and equals zero if and only if $\nu(s_1 = 0) = 0$ (as we have suppose (3); see p.28 [4]).

After these definitions, we introduce a fundamental martingale associated to $(\xi_u, u \in \mathcal{U})$.

Theorem 1. *The process*

$$M_n := \sum_{|u|=n} \xi_u^{p_0}, \quad n \in \mathbb{N}$$

is a martingale in the filtration $(\mathcal{H}_{L_n})$, with L_n the line associated to the n-th generation (i.e. $L_n := \{u \in \mathcal{U} : \ |u| = n\}$). This martingale is bounded in $\mathrm{L}^p(\mathbb{P})$ for some $p > 1$, and in particular is uniformly integrable.

Moreover, conditionally on non-extinction the terminal value M_∞ is strictly positive a.s.

Remark 1. *As κ is concave the equation $\kappa(p) = 0$ may have a second root $p_+ := \inf\{p > p_0,\ \kappa(p) = 0\}$). This second root is less interesting: even though*

$$M_n^+ := \sum_{|u|=n} \xi_u^{p_+}, \quad n \in \mathbb{N},$$

is also a martingale, it is easy to check that for all $p > 1$ the p-variation of M_n^+ is infinite, i.e. $\mathbb{E}\left(\sum_{n=0}^\infty |M_{n+1} - M_n|^p\right) = \infty$).

We can notice that for all $p \in (p_0, p_+)$ $(M_n^{(p)})_{n \in \mathbb{N}} := (\sum_{|u|=n} \xi_u^{p})_{n \in \mathbb{N}}$ is a supermartingale.

The assumption (3) means actually that $\mathbb{E}(M_1^p) < \infty$.

Proof. • We will use the fact that the empirical measure of the logarithm of the sizes of fragments

$$Z^{(n)} := \sum_{|u|=n} \delta_{\log \xi_u} \tag{4}$$

can be viewed as a branching random walk (see the article of Biggins [8]) and use Theorem 1 of [8]. In order to do that we first introduce some notation: for $\theta > \underline{p}$, we define

$$m(\theta) := \mathbb{E}\left(\int e^{\theta x} Z^{(1)}(dx)\right) = \mathbb{E}\left(\sum_{|u|=1} \xi_u^\theta\right) = 1 - \kappa(\theta)$$

and

$$W^{(n)}(\theta) := m(\theta)^{-n} \int e^{\theta x} Z^{(n)}(dx) = (1-\kappa(\theta))^{-n} \sum_{|u|=n} \xi_u^\theta.$$

We notice that $M_n = W^{(n)}(p_0)$. Therefore in order to apply Theorem 1 of [8] and to get the convergence almost surely and in pth mean for some $p > 1$, it is enough to show that

$$\mathbb{E}(W^{(1)}(p_0)^\gamma) < \infty$$

for some $\gamma \in (1, 2]$ and

$$m(pp_0)/|m(p_0)|^p < 1$$

for some $p \in (1, \gamma]$. The first condition is a consequence of the Malthusian assumption. Moreover the second follows from the identities

$$m(pp_0)/|m(p_0)|^p = (1-\kappa(pp_0))/|1-\kappa(p_0)|^p = 1 - \kappa(pp_0)$$

which, by the definition of p_0, is smaller than 1 for $p > 1$ well chosen.

• Finally, let us now check that $M_\infty > 0$ *a.s.* conditionally on non-extinction. Define $q = \mathbb{P}(M_\infty = 0)$, therefore as $\mathbb{E}(M_\infty) = 1$ we get that $q < 1$. Moreover, an application of the branching property yields

$$\mathbb{E}(q^{Z_n}) = q,$$

where Z_n is the number of individuals with positive size at the n-th generation. Notice that $Z_n = \langle Z^{(n)}, 1\rangle$. By the construction of the marked tree and as ν is a probability measure: $(Z_n, n \in \mathbb{N})$ is of course a Galton-Watson process and it follows that q is its probability of extinction. Since $M_\infty = 0$ conditionally on the extinction, the two events coincide a.s. □

4 Evolution of the process in continuous time.

After having defined the process indexed by generation and having shown that the martingale M_n is $\mathrm{L}^p(\mathbb{P})$ bounded, we are now able to define properly the main objet of this paper. In order to do this, when an individual labelled by u has a positive size, $\xi_u > 0$, let $I_u := [a_u, a_u + \zeta_u)$ be the interval of times during which this individual is alive. Otherwise, i.e. when $\xi_u = 0$, we decide that $I_u = \emptyset$. With this definition, we set:

Definition 2. *We define the process* $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ *by*

$$\mathbf{X}(t) = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u\}} \delta_{\xi_u}, t \geq 0. \tag{5}$$

In particular we have for $f : \mathbb{R}_+ \to \mathbb{R}$ *measurable function*

$$\langle f, \mathbf{X}(t) \rangle = \sum_{u \in \mathcal{U}} f(\xi_u) \mathbb{1}_{\{t \in I_u\}}.$$

For every $x > 0$, let $\mathbb{P}_x$ be the law of the process $\mathbf{X}$ starting from a single individual with size x. And for simplification, we denote $\mathbb{P}$ for $\mathbb{P}_1$, and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of the process $(\mathbf{X}(t), t \geq 0)$. We use the notation $(X_1(t), ..., X_{\sharp \mathbf{X}(t)}(t))$ for the sequence of atoms of $\mathbf{X}(t)$. In the following we will show that this sequence is almost surely finite. Of course the set $(X_1(t), ..., X_{\sharp \mathbf{X}(t)}(t))$ is the same as the set $((\xi_u); t \in I_u)$; but sometimes it will be clearer to use the notation $(X_i(t))$.

We define for $u \in \mathbb{R}_+$:

$$F(u) := \int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{\sharp \mathbf{s}} \nu(d\mathbf{s}).$$

We notice that $F(u)$ is the generating function of the Galton-Watson process $(Z_n, n \geq 0) = (\sharp\{u \in \mathcal{U} : \ \xi_u > 0 \text{ and } \ |u| = n\}, n \geq 0)$.

From now on, we will suppose that for every $\epsilon > 0$

$$\int_{1-\epsilon}^{1} \frac{du}{F(u) - u} = \infty. \tag{6}$$

Of course if $F^{'}(1) = \mathbb{E}(Z_1) < \infty$ this last assumption is fulfilled. Therefore we get the first theorem about the continous time process:

Theorem 2. *The process* $\mathbf{X}$ *takes its values in the set* $\mathcal{M}_p(\mathbb{R}_+^*)$. *It is a branching Markov chain, more precisely the conditional distribution of* $\mathbf{X}(t+r)$ *given that* $\mathbf{X}(r) = \mathbf{s}$ *is the same as that of the sum* $\sum \mathbf{X}^{(i)}(t)$, *where for each index* i, $\mathbf{X}^{(i)}(t)$ *is distributed as* $\mathbf{X}(t)$ *under* $\mathbb{P}_{s_i}$ *and the variables* $\mathbf{X}^{(i)}(t)$ *are independent.*

The process $\mathbf{X}$ *also has the scaling property, namely for every* $c > 0$, *the distribution of the rescaled process* $(c\mathbf{X}(c^{\alpha} t), t \geq 0)$ *under* $\mathbb{P}_1$ *is* $\mathbb{P}_c$.

In the fragmentation case, the fact that the size of the fragments decreases with time entails that the process of the fragments of size larger than or equal to ϵ is Markovian, and which leads easily to Theorem 2. This property is lost in the present case.

Proof. • First we will check that for all $t \geq 0$, $\mathbf{X}(t)$ is a (random) finite point measure. By Theorem 1 and the Doob's L^p-inequality we get that for some $p > 1$:

$$\sup_{n \in \mathbb{N}} M_n = \sup_{n \in \mathbb{N}} \sum_{|u|=n} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P}).$$

As a consequence:

$$\sup_{u\in\mathcal{U}} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P})$$

and then by the definition of the process $\mathbf{X}$, writing $X_1(t), \ldots$ for the (possibly infinite) sequence of atoms of $\mathbf{X}(t)$

$$\sup_i \sup_{t\in\mathbb{R}_+} X_i(t)^{p_0} \in \mathrm{L}^p(\mathbb{P}).$$

Recall that $p_0 > 0$ by assumption. We fix some arbitrarily large $m > 0$. We now work conditionally on the event that the size of all individuals is bounded by m, and we will show that the number of the individuals alive at time t is almost surely finite for all $t \geq 0$.

As we are conditioning on the event $\{\sup_{u\in\mathcal{U}} \xi_u \leq m\}$, by the construction of the marked tree, we get that the life time of an individual can be stochastically bounded from below by an exponential variable of parameter m^α. Therefore we can bound the number of individuals present at time t by the number of individuals of a continuous time branching process denoted by GW in which each individual lives for a random time whose law is exponential of parameter m^α and the probability distribution of the offspring is the law of $\sharp s \vee 1$ under ν (we have taken the supremum with 1 to ensure the absence of death). For the Markov branching process GW, we are in the temporally homogeneous case and, we notice that

$$\int_{\mathcal{M}_p(\mathbb{R}_+^*)} u^{(n_\mathbf{s})\vee 1} \nu(d\mathbf{s}) = (f(u) - u)\nu(n_\mathbf{s} \neq 0) + u,$$

therefore as we have supposed (6), we can use Theorem 1 p.105 of the book of Athreya and Ney [3] (proved in Theorem 9 p.107 of the book of Harris [14]) and get that we are in the non-explosive case for the GW. As the number of the individuals is bounded by that of GW we get that the number of individuals at time t is a.s. finite.

Therefore conditioning on the event $\{\sup_{u\in\mathcal{U}} \xi_u \leq m\}$, we have that for all $t \geq 0$, the number of individuals at time t is a.s. finite, i.e. $\mathbf{X}(t)$ is a finite point measure.

• Second we will show the Markov property. Fix $r \in \mathbb{R}_+$. Let τ_r be equal to $\{u \in \mathcal{U} : r \in I_u\}$. We notice that τ_r is an optional line. In fact for all lines $L \subset \mathcal{U}$ we have that

$$\{\tau_r \preceq L\} = \{r < a_u + \zeta_u \quad \forall u \in L\} \in \mathcal{H}_L.$$

By definition, we have the identity

$$\sum_{i=1}^{\sharp\mathbf{X}(t+r)} \mathbb{1}_{\{X_j(t+r)>0\}} \delta_{X_j(t+r)} = \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t+r\in I_u\}} \delta_{\xi_u}.$$

Let $\mathbf{X}(r) = \sum_{i=1}^n \delta_{\xi_{v_n}} \in \mathcal{M}_p(\mathbb{R}_+^*)$ with $n = \sharp\mathbf{X}(r)$ and $(v_1, \ldots, v_n)$ the nodes of $\mathcal{U}$. Define for all $i \leq n$,

$$\tilde{T}^{(i)} := ((\xi_{v_i u}, a_{v_i u} - a_{v_i}, \zeta_{v_i u} - \mathbb{1}_{\{u=\emptyset\}}(r - a_{v_i}))_{u\in\mathcal{U}}) = ((\tilde{\xi}_u^{(i)}, \tilde{a}_u^{(i)}, \tilde{\zeta}_u^{(i)})_{u\in\mathcal{U}}),$$

$\tilde{I}_u^{(i)} := [\tilde{a}_u^{(i)}, \tilde{a}_u^{(i)} + \tilde{\zeta}_u^{(i)}[$ and

$$\mathbf{X}^{(i)}(t) = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in \tilde{I}_u^{(i)}\}} \delta_{\tilde{\xi}_u^{(i)}}.$$

Then

$$\mathbf{X}(t+r) = \sum_{i=1}^{n} \mathbf{X}^{(i)}(t).$$

By the lack of memory of the exponential variable, we have that for $u \in \mathcal{U}$, given $s \in I_u$ the law of the marked tree $\tilde{T}^{(i)}$ is the same as that of

$$T^{\xi_{v_i}} := ((\xi_{v_i u}, a_{v_i u} - a_{v_i}, \zeta_{v_i u})_{u \in \mathcal{U}}) := ((\xi_u^i, a_u^i, \zeta_u^i)_{u \in \mathcal{U}}).$$

Thus we have the equality in law:

$$\sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in \tilde{I}_u^{(i)}\}} \delta_{\tilde{\xi}_u^{(i)}} \overset{(d)}{=} \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u^i\}} \delta_{\xi_u^i},$$

with $I_u^i := [a_u^i, a_u^i + \zeta_u^i[$.

Let $\tau_r^i := \{v_i u \in \mathcal{U} : \ r \in I_u^i\}$. Moreover for all lines $L \in \mathcal{U}$ we have that

$$\{\tau_r^i \preceq L\} = \{r < a_{v_i u} + \zeta_{v_i u} \quad \forall v_i u \in L\} \in \mathcal{H}_L.$$

Therefore τ_r^i is an optional line and by applying Lemma 1 for the optional line τ_s^i, we have that the condition distribution of the point measure

$$\sum_{u \in \mathcal{U}} \mathbb{1}_{\{t+r \in I_u^i\}} \delta_{\xi_u^i}$$

given $\mathcal{H}_{\tau_r}$ is the law of $\mathbf{X}(t)$ under $\mathbb{P}_{x_i}$. We notice that $\mathcal{H}_{\tau_s} = \sigma(\tilde{\xi}_u, \mathbf{e}_u : a_u \leq s)$ is the same filtration as $\mathcal{F}_s = \sigma(\mathbf{X}(s') : \ s' \leq s)$. Therefore $(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, ..., \mathbf{X}^{(n)})$ is a sequence of independent random processes, where for each i $\mathbf{X}^{(i)}(t)$ is distributed as $\mathbf{X}(t)$ under $\mathbb{P}_{x_i}$. We then have proven the Markovian property.

- The scaling property is an easy consequence of the definition of the tree T_x. $\square$

Remark 2. *For every measurable function $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$, define a multiplicative functional such that for every $\mathbf{s} = \sum_{i=1}^{\sharp s} \delta_{s_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$:*

$$\phi_g(\mathbf{s}) := \exp(-\langle g, \mathbf{s} \rangle) = \exp(-\sum_{i=1}^{\sharp s} g(s_i)).$$

Then the generator G of the Markov process $\mathbf{X}(t)$ fulfills for every $\mathbf{y} = \sum_{i=1}^{\sharp \mathbf{y}} \delta_{y_i} \in \mathcal{M}_p(\mathbb{R}_+^*)$:

$$G\phi_g(\mathbf{y}) = \sum y_i^{\alpha} e^{-\sum_{j \neq i} g(y_j)} \int_{\mathcal{M}_p(\mathbb{R}_+^*)} (e^{-\langle g(xy_i), \mathbf{s} \rangle} - e^{-g(y_i)}) \nu(d\mathbf{s}). \tag{7}$$

The intrinsic martingale M_n is indexed by the generations; it will also be convenient to consider its analogue in continuous time, i.e

$$M(t) := \langle x^{p_0}, \mathbf{X}(t)\rangle = \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t\in I_u\}}\xi_u^{p_0}.$$

It is straightforward to check that $(M(t), t \geq 0)$ is again a martingale in the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process $(\mathbf{X}(t), t \geq 0)$; and more precisely, the argument Proposition 1.5 in [4] gives:

Corollary 3. *The process $(M(t), t \geq 0)$ is a martingale, and more precisely*

$$M(t) = \mathbb{E}(M_\infty|\mathcal{F}_t),$$

where M_∞ is the terminal value of the intrinsic martingale $(M_n, n \in \mathbb{N})$. In particular $M(t)$ converges in $\mathrm{L}^p(\mathbb{P})$ to M_∞ for some $p > 1$.

Proof. We will use the same argument as in the proof of Proposition 1.5 of [4]. Netherless, we have to deal here with the fact that $\sup_{u\in\mathcal{U}} \xi_u$ may be larger than 1. Therefore we will have to condition. We know that M_n converges in $\mathrm{L}^p(\mathbb{P})$ to M_∞ as n tends to ∞, so

$$\mathbb{E}(M_\infty|\mathcal{F}_t) = \lim_{n\to\infty} \mathbb{E}(M_n|\mathcal{F}_t).$$

By Theorem 1 as we have

$$\sup_{u\in\mathcal{U}} \xi_u^{p_0} \in \mathrm{L}^p(\mathbb{P}),$$

we fix $m > 0$. We now work on the event $B_m := \{\sup_{u\in\mathcal{U}} \xi_u \leq m\}$.

By applying the Markov property at time t we easily get that

$$\mathbb{E}(M_n|\mathcal{F}_t) = \sum_{i=1}^{\sharp\mathbf{X}(t)} X_i^{p_0}(t)\mathbb{1}_{\{\varrho(X_i(t))\leq n\}} + \sum_{|u|=n} \xi_u^{p_0}\mathbb{1}_{\{a_u+\zeta_u<t\}} \tag{8}$$

where $\varrho(\xi_v)$ stands for the generation of the individual v (i.e. $\varrho(\xi_v) = |v|$), and $a_u + \zeta_u$ is the instant when the individual corresponding to the node u reproduces. We can rewrite the latter as

$$a_u + \zeta_u = \xi_{m^{|u|}u}^{-\alpha}\mathbf{e}_0 + \xi_{m^{|u|-1}u}^{-\alpha}\mathbf{e}_1 + ... + \xi_u^{-\alpha}\mathbf{e}_{|u|}$$

where $\mathbf{e}_0$,... is a sequence of independent exponential variables with parameter 1, which is also independent of ξ_u. We can remark that in the first term of sum (8) we sum over the sizes of the individuals which belong to the n-th generation and are alive at time t, and in the second term we sum over those belonging to the n-th generation and are dead at time t.

As α is nonnegative, and as we are working on the event B_m: $\xi_{m^i u}^{-\alpha} \geq m^{-\alpha}$ we have that for each fixed node $u \in \mathcal{U}$, $a_u + \zeta_u$ is bounded from below by the sum of $|u| + 1$ independent exponential variables with parameter m^α which are independent of ξ_u. Thus

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{|u|=n} \xi_u^{p_0}\mathbb{1}_{\{a_u+\zeta_u<t\}}\mathbb{1}_{\{B_m\}}\right) = 0,$$

and therefore by (8) on the event $\{B_m\}$, we get that for all $m > 0$: $\mathbb{E}(M_\infty|\mathcal{F}_t)\mathbb{1}_{\{B_m\}} = M(t)\mathbb{1}_{\{B_m\}}$, and then by letting m tend to ∞ we get the result. $\square$

5 A randomly tagged leaf.

We will here (as in [4]) define what a tagged individual is by using a tagged leaf.

We call *leaf* of the tree $\mathcal{U}$ an infinite sequence of integers $l = (u_1, ...)$. For each n, $l^n := (u_1, ..., u_n)$ is the ancestor of l at the generation n. We enrich the probabilistic structure by adding the information about a so called tagged leaf, chosen at random as follows. Let H_n be the space of bounded functionals Φ which depend on the mark M and of the leaf l up to the n-th first generation, i.e. such that $\Phi(M, l) = \Phi(M', l')$ if $l^n = l^{n'}$ and $M(u) = M'(u)$ whenever $|u| \le n$. For such functionals, we use the slightly abusing notation $\Phi(M, l) = \Phi(M, l^n)$. As in [4] for a pair (M, λ) where $M : \mathcal{U} \to [0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a random mark on the tree and λ is a random leaf of $\mathcal{U}$, the joint distribution denoted by $\mathbb{P}^*$ (and by $\mathbb{P}^*_x$ if the size of the first mark is x instead of 1) can be defined unambiguously by

$$\mathbb{E}^*(\Phi(M, \lambda)) = \mathbb{E}\left(\sum_{|u|=n} \Phi(M, u)\xi_u^{p_0}\right), \qquad \Phi \in \mathrm{H}_n.$$

Moreover since the intrinsic martingale $(M_n, n \in \mathbb{Z}_+)$ is uniformly integrable (cf. Theorem 1), the first marginal of $\mathbb{P}^*$ is absolutely continuous with respect to the law of the random mark M under $\mathbb{P}$, with density M_∞.

Let λ_n be the node of the tagged leaf at the n-th generation. We denote $\chi_n := \xi_{\lambda_n}$ for the size of the individual corresponding to the node λ_n and $\chi(t)$ for the size of the tagged individual alive at time t, viz.

$$\chi(t) := \chi_n \quad \text{ if } \; a_{\lambda_n} \le t < a_{\lambda_n} + \zeta_{\lambda_n},$$

because in the case considered $\sup_{n\in\mathbb{N}} a_{\lambda_n} = \infty$. We stress that, in general the process $\chi(t)$ is not monotonic. However as in [4], Lemma 1.4 there becomes:

Lemma 2. *Let $k : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function such that $k(0) = 0$. Then we have for every $n \in \mathbb{N}$*

$$\mathbb{E}^*(k(\chi_n)) = \mathbb{E}\left(\sum_{|u|=n} \xi_u^{p_0} k(\xi_u)\right),$$

and for every $t \ge 0$

$$\mathbb{E}^*(k(\chi(t))) = \mathbb{E}\left(\langle x^{p_0} k(x), X(t)\rangle\right).$$

Proposition 1.6 of [4] becomes:

Proposition 4. *Under $\mathbb{P}^*$,*

$$S_n := \ln \chi_n, \quad n \in \mathbb{Z}_+$$

is a random walk on $\mathbb{R}$ with step distribution

$$\mathbb{P}(\ln \chi_n - \ln \chi_{n+1} \in dy) = \widetilde{\nu}(dy),$$

where the probability measure $\widetilde{\nu}$ is defined by

$$\int_{]0,\infty[} k(y)\widetilde{\nu}(dy) = \int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} k(\ln(x)), \mathbf{s}\rangle \nu(d\mathbf{s}).$$

Equivalently, the Laplace transform of the step distribution is given by

$$\mathbb{E}^*(e^{pS_1}) = \mathbb{E}^*(\chi_1^p) = 1 - \kappa(p + p_0), \quad p \geq 0.$$

Moreover, conditionally on $(\chi_n, n \in \mathbb{Z}_+)$ the sequence of the lifetimes $(\zeta_{\lambda_0}, \zeta_{\lambda_1}, ...)$ along the tagged leaf is a sequence of independent exponential variables with respective parameters $\chi_0^\alpha, \chi_1^\alpha, ...$

We now see that we can use this proposition to obtain the description of $\chi(t)$ using a Lamperti transformation. Let

$$\eta_t := S \circ N_t, \quad t \geq 0,$$

with N a Poisson process with parameter 1 which is independent of the random walk S; for probabilities and expectations related to η we use the notation P and E. The process $(\chi(t), t \geq 0)$ is Markovian and enjoys a scaling property. More precisely under $\mathbb{P}_x^*$ we get that

$$\chi(t) \overset{(d)}{=} \exp(\eta_{\tau(tx^{-\alpha})}), \quad t \geq 0, \tag{9}$$

where η is the compound Poisson defined above and τ the time-change defined implicitly by

$$t = \int_0^{\tau(t)} \exp(\alpha \eta_s) ds, \quad t \geq 0. \tag{10}$$

6 Asymptotic behaviors.

6.1 The convergence of the size of a tagged individual.

Let

$$\kappa^{'}(p_0) = -\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0} \ln(x), \mathbf{s}\rangle \nu(d\mathbf{s})$$

denote the derivative of κ at the Malthusian parameter p_0.

In this part we focus on the asymptotic behavior of the size of a tagged individual. In this direction, the quantity $\varpi_t = e^{\alpha \eta_t}$ plays an important role, as it appears at the time change of the Lamperti transformation (see (10)), as we see in the next proposition:

Proposition 5. *Suppose that $\alpha > 0$, that the support of ν is not a discrete subgroup $r\mathbb{Z}$ for any $r > 0$ and that $0 < \kappa^{'}(p_0) < \infty$. Then for every $y > 0$, under $\mathbb{P}_y^*$, $t^{1/\alpha}\chi(t)$ converges in law as $t \to \infty$ to a random variable Y whose law is specified by*

$$\mathbb{E}(k(Y^\alpha)) = \frac{1}{\alpha m_1} E(k(I)I^{-1}),$$

for every measurable function $k : \mathbb{R}_+ \to \mathbb{R}_+$, with $I := \int_0^\infty \exp(\alpha\eta_s)ds$ and $m_1 := E(\eta_1) = -\kappa^{'}(p_0)$.

Proof. As $-\kappa^{'}(p_0)$ is the mean of the step distribution of the random walk S_n (see Proposition 4), therefore $\kappa^{'}(p_0) > 0$ imply that $E(-\eta_1) > 0$ thus the assumption of Theorem 1 in the works of Bertoin and Yor [7] is fulfilled by the self-similar Markov process $\chi(t)^{-1}$, which gives the result. □

We could also try to use the same method as the one used in [6] for which we need Proposition 1.7 [4]. But in this latter we needed $\mathbb{E}(\langle x^p, X(t)\rangle)$ to be finite when p is large, and its derivative to be completely monotone. But here neither of these requirements is necessarily true as κ is not necessarily positive when p is large. This explains why we have to use a different method.

Remark 3. *In the case $\kappa^{'}(p_0) = 0$ we can extend this proposition. More precisely if $\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \langle x^{p_0}|\ln(x)|, \mathbf{s}\rangle \nu(d\mathbf{s}) < \infty$,*

$$J := \int_1^\infty \frac{x\nu^-((x,\infty))dx}{1 + \int_0^x dy \int_y^\infty \nu^-((-\infty,-z))dz} < \infty,$$

(where ν^- is the image of $\widetilde{\nu}$ by the map $u \to -u$ and $\widetilde{\nu}$ is defined in Proposition 4) and $E\left(\log^+ \int_0^{T_1} e^{-\eta_s} ds\right) < \infty$ (with $T_z := \inf\{t : -\eta_t \geq z\}$) hold then, for any $y > 0$ under $\mathbb{P}_y^*$, $t^{1/\alpha}\chi(t)$ converge in law as $t \to \infty$, to a random variable $\tilde{Y}$ whose law is specified by

for any bounded and continuous function k and for $t > 0$:

$$\mathbb{E}(k(\tilde{Y}^\alpha)) = \lim_{\lambda \to 0} \frac{1}{\lambda} E(I_\lambda^{-1} k(I_\lambda)),$$

where $I_\lambda = \int_0^\infty \exp(\alpha\eta_s - \lambda s)ds$.

The proof is the same as the previous one using Theorem 1 and Theorem 2 from the works of Caballero and Chaumont [11] instead of [7].

6.2 Convergence of the mean measure and L^p-convergence.

We encode the configuration of masses $X(t) = \{(X_i(t))_{1 \leq i \leq \sharp\mathbf{X}(t)}\}$ by the weighted empirical measure

$$\sigma_t := \sum_{i=1}^{\sharp\mathbf{X}(t)} X_i^{p_0}(t)\delta_{t^{1/\alpha}X_i(t)}$$

which has total mass $M(t)$.

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty k(x)\sigma_t^*(dx) = \mathbb{E}\left(\int_0^\infty k(x)\sigma_t(dx)\right)$$

which is required to hold for all compactly supported continuous functions k. Since $M(t)$ is a martingale, σ_t^* is a probability measure. We interest us to the convergence of this measure. This convergence was already established in the case of binary conservative fragmentation (see the results of Brennan and Durrett [9] and [10]). A very useful tool for this is the renewal theorem, for which they needed the fact that the process $\chi(t)$ is decreasing; here we no longer have such a monotonicity property. See also Theorem 2 and 5 of [6], Theorem 1.3 of [4] and Proposition 4 of [17] for Theorems about empirical measure for measure which have a conservative property $\nu(s_i \leq 1\ \forall i \in \mathbb{N}) = 1$.

Nonetheless, with Proposition 5 and Lemma 2, we easily get:

Corollary 6. *With the assumptions of Proposition 5 we get:*

1. *The measures σ_t^* converge weakly, as $t \to \infty$, to the distribution of Y i.e. for any continuous bounded function $k : \mathbb{R}_+ \to \mathbb{R}_+$, we have:*

$$\mathbb{E}\left(\langle x^{p_0} k(t^{1/\alpha}x), X(t)\rangle\right) \underset{t\to\infty}{\rightarrow} \mathbb{E}(k(Y)).$$

2. *For all $p_+ > p > p_0$:*

$$t^{(p-p_0)/\alpha}\mathbb{E}\left(\langle x^p, X(t)\rangle\right) \underset{t\to\infty}{\rightarrow} \mathbb{E}(Y^{p-p_0}).$$

We now formulate a more precise result concerning the convergence of the empirical measure:

Theorem 7. *Under the same assumptions as in Proposition 5 we get that for every bounded continuous function k:*

$$\mathrm{L}^p - \lim_{t\to\infty} \int_0^\infty k(x)\sigma_t(dx) = M_\infty \mathbb{E}(k(Y)) = \frac{M_\infty}{\alpha m} E(k(I)I^{-1}),$$

for some $p > 1$.

Remark 4. *A slightly different version of Corollary 6 and Theorem 7 exists also under the assumptions in Remark 3.*

See also Asmussen and Kaplan [1] and [2] for a closely related result.

Proof. We follow the same method as Section 1.4. in [4] and in this direction we use Lemma 1.5 there: for $(\lambda(t))_{t\geq 0} = (\lambda_i(t), i \in \mathbb{N})_{t\geq 0}$ a sequence of non-negative random variables such that for fixed $p > 1$

$$\sup_{t\geq 0} \mathbb{E}\left(\left(\sum_{i=1}^{\infty} \lambda_i(t)\right)^p\right) < \infty \quad \text{and} \quad \lim_{t\to\infty} \mathbb{E}\left(\sum_{i=1}^{\infty} \lambda_i(t)\right) = 0,$$

and for $(Y_i(t), i \in \mathbb{N})$ a sequence of random variables which are independent conditionally on $\lambda(t)$, we assume that there exists a sequence $(\bar{Y}_i, i \in \mathbb{N})$ of i.i.d variables in $\mathrm{L}^p(\mathbb{P})$, which is independent of $\lambda(t)$ for each fixed t, and such that $|Y_i(t)| \leq \bar{Y}_i$ for all $i \in \mathbb{N}$ and $t \geq 0$.

Then we know from Lemma 1.5 in [4] that

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}\left(Y_i(t)|\lambda(t)\right)) = 0. \tag{11}$$

Now, let k be a continuous function bounded by 1 and let

$$A_t := \langle x^{p_0} k(t^{1/\alpha} x), X(t)\rangle.$$

By application of the Markov property at time t for A_{t+s} and the self-similarity property of the process $\mathbf{X}$ we can rewrite A_{t+s} as

$$\sum_{i=1}^{\sharp \mathbf{X}(t)} \lambda_i(t) Y_i(t,s)$$

where $\lambda_i(t) := X_i^{p_0}(t)$ and

$$Y_i(t,s) := \langle x^{p_0} k((t+s)^{1/\alpha} X_i(t) x), \mathbf{X}_{i,.}(s)\rangle,$$

with $\mathbf{X}_{1,.}$, $\mathbf{X}_{2,.}$, ... a sequence of i.i.d. copies of $\mathbf{X}$ which is independent of $\mathbf{X}(t)$.

By Theorem 1 we get that

$$\sup_{t\geq 0} \mathbb{E}\left(\left(\sum_{i=1}^{\sharp \mathbf{X}(t)} \lambda_i(t)\right)^p\right) < \infty.$$

By the last corollary we also obtain that

$$\mathbb{E}\left(\sum_{i=1}^{\sharp \mathbf{X}(t)} \lambda_i^p(t)\right) \sim t^{-(p-1)p_0} \mathbb{E}(\chi^{(p-1)p_0}(1)) \to 0,$$

as $t \to \infty$.

Moreover the variables $Y_i(t,s)$ are uniformly bounded by

$$Y_i = \sup_{s\geq 0} \langle x^{p_0}, \mathbf{X}_{i,.}(s)\rangle,$$

which are i.i.d. variables and also bounded in $\mathrm{L}^p(\mathbb{P})$ thanks to Doob's inequality (as $\langle x_{p_0}, \mathbf{X}_{i,.}(s)\rangle$ is a martingale bounded in $\mathrm{L}^p(\mathbb{P})$).

Thus we may apply (11), which reduces the study to that of the asymptotic behavior of:

$$\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)\mathbb{E}(Y_i(t,s)|\mathbf{X}(t)),$$

as t tends to ∞. On the event $\{X_i(t) = y\}$, we get

$$\mathbb{E}(Y_i(t,s)|\mathbf{X}(t)) = \mathbb{E}\left(\langle x^{p_0} k((t+s)^{1/\alpha} yx), \mathbf{X}(s)\rangle\right).$$

Then by Lemma 2:

$$\mathbb{E}\left(\langle x^{p_0} k((t+s)^{1/\alpha} yx), \mathbf{X}_{i,.}(s)\rangle\right) = \mathbb{E}^*\left(k\left((t+s)^{1/\alpha} y\chi(s)\right)\right).$$

With Proposition 5, we obtain

$$\lim_{t\to\infty} \mathbb{E}^*\left(k\left((t+s)^{1/\alpha} y\chi(s)\right)\right) = \mathbb{E}\left(k\left(Y\right)\right).$$

Moreover recall from Corollary 3 that $\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)$ converges to M_∞ in $\mathrm{L}^p(\mathbb{P})$. Therefore we finally get that when t goes to infinity:

$$\sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t)\mathbb{E}(Y_i(t,s)|X(t)) \sim \mathbb{E}\left(k\left(Y\right)\right) \sum_{i=1}^{\sharp\mathbf{X}(t)} \lambda_i(t) \sim \mathbb{E}\left(k\left(Y\right)\right) M_\infty.$$

□

A Further results about the intrinsic process

We will give more general properties about the intrinsic process $\{M_Q,\ Q \subset \mathcal{U}\}$, $M_Q = \sum_{u\in M} \xi_u^{p_0}$. For a line Q, $\{M_Q\}$ is adapted to the filtration $\{\mathcal{H}_L\}$. We use the abuse of notation that M_n stand for the process M_{L_n}, with $L_n = \{u \in \mathcal{U} :\ |u| = n\}$ the labels of the n-th generation. We introduce new definitions, we say that a line Q *covers* L, if $Q \succeq L$ and any individual stemming from L either stems from Q or has progeny in Q. If Q covers the ancestor it may simply be called *covering*. Let $\mathcal{C}_0$ be the class of covering lines with finite maximal generation. We denoted the generation of Q: $|Q| = \sup_{u\in Q} |u|$. The origin of the intrinsic martingale comes from real time martingale of Nerman [20].

Also for $r \in \mathbb{R}_+^*$, let ϑ_r be the structural measure:

$$\vartheta_r(B) := \mathbb{E}_r(\sharp\{u \in \mathcal{U} : \xi_u \in B\}) = \sum_{i=1}^{\infty} \nu(rs_i \in B) \quad \text{for } B \subset \mathcal{B},$$

where $\mathcal{B}$ is the Borel algebra on $\mathbb{R}_+^*$. Let the reproduction measure μ on the sigma-field $\mathcal{B} \otimes \mathcal{B}$ be such that for every $r \geq 0$:

$$\mu(r, dv \times du) := r^{\alpha} e^{-r^{\alpha} u} du \sum_{i=1}^{\infty} \nu(rs_i \in dv)$$

and for any $\lambda \in \mathbb{R}$

$$\mu_\lambda(r, dv \times du) := e^{-\lambda u} \mu(r, du \times dv).$$

The composition operation $*$ denotes the Markov transition on the size space $\mathbb{R}_+$ and convolution on the time space $\mathbb{R}_+$, so that: for all $A \in \mathcal{B}$ and $B \in \mathcal{B}$,

$$\mu^{*2}(s, A \times B) = \mu * \mu(s, A \times B) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \mu(r, A \times (B - u)) \mu(s, dr \times du).$$

With the convention that the $*$-power 0 is $\mathbb{1}_{\{A \times B\}}(s, 0)$ which gives all the mass to $(s, 0)$. We define the renewal measure as

$$\psi_\lambda := \sum_{0}^{\infty} \mu_\lambda^{*n}.$$

Let

$$\alpha' := \inf\{\lambda \ : \ \ \psi_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) < \infty \ \text{ for some } \ r \in \mathbb{R}_+\}.$$

Moreover as

$$\mu_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) = \begin{cases} mr^{\alpha}/(r^{\alpha} + \lambda) & \text{if } \ \lambda > -r^{\alpha} \\ \infty & \text{else,} \end{cases}$$

thus

$$\psi_\lambda(r, \mathbb{R}_+ \times \mathbb{R}_+) < \infty \ \text{ if and only if } \lambda < (r/(m-1))^{1/\alpha}$$

therefore we get $\alpha' = 0$. For $A \in \mathcal{B}$, let

$$\pi(A) := \lim_{n \to \infty} \mu^{*n}(1, A \times \mathbb{R}_+) \tag{12}$$

which is well defined as $\mu^{*n}(1, A \times \mathbb{R}_+)$ is a decreasing function in n and nonnegative. Let $h(s) := s^{p_0}$ for all $s \in \mathbb{R}_+$ and $\beta := 1$. These objects correspond to those defined in [15].

Recall that the Galton-Watson process $(Z_n, n \geq 0))$ is equal to $(\sharp\{u \in \mathcal{U} \ : \ \ \xi_u > 0 \text{ and } \ |u| = n\}, n \geq 0)$.

We suppose that

$$m := \mathbb{E}(Z_1) < \infty,$$

i.e. $\int_{\mathcal{M}_p(\mathbb{R}_+^*)} \sharp s \nu(d\mathbf{s}) < \infty$ this assumption is slightly stronger than (6), therefore we get that:

Proposition 8. *1. If $L \preceq Q$ are lines, then*

$$\mathbb{E}(M_Q|\mathcal{H}_L) \leq M_L.$$

If Q verifies $|Q| < \infty$ and covers L, then

$$\mathbb{E}(M_Q|\mathcal{H}_L) = M_L.$$

2. For all $s > 0$, $\{M_L;\ L \in \mathcal{C}_0\}$ is uniformly $\mathbb{P}_s$-integrable.

3. There is a random variable $M \geq 0$ such that for π-almost all $s > 0$

$$M_L = \mathbb{E}_s(M|\mathcal{H}_L)$$

and $M_L \overset{L^1(\mathbb{P}_s)}{\to} M$, as $L \in \mathcal{C}_0$ filters $(\preceq)$. If $\varsigma_n \preceq \varsigma_{n+1} \in \mathcal{C}_0$ and to any $x \in \mathcal{U}$ there is an ς_n such that x has progeny in ς_n, $M_{\varsigma_n} \to M$, as $n \to \infty$, also a.s. $\mathbb{P}_s$.

A consequence of the first and second points applied for $L_n = \{u \in \mathcal{U} :\ |u| = n\}$ and $L_m = \{u \in \mathcal{U} :\ |u| = m\}$ with $m \geq n \geq 0$, is that M_n is a martingale and the uniform $\mathbb{P}_s$-integrability of this martingale. The third point applied for the lines τ_t give the convergence of $M(t)$ in $\mathrm{L}^1(\mathbb{P}_s)$ and almost surely.

Proof. • First the conditions of Malthusian population are fulfilled, thus by Theorem 5.1 in [15] we get the first point.

Let $\overline{\xi} := \int_{\mathbb{R}_+\times\mathbb{R}_+} h(s) r^\alpha e^{-tr^\alpha} dt \vartheta_1(ds) = \sum_{|u|=1} \xi_u^{p_0}$ and $\mathbb{E}_\pi$ be the expectation with respect to $\int_{\mathbb{R}_+} \mathbb{P}_s(dw)\pi(ds)$. Therefore,

$$\mathbb{E}_\pi(\overline{\xi}\log^+\overline{\xi}) = \int_{\mathbb{R}_+} \mathbb{E}_x\left(\sum_{i=1}^{\infty} \xi_i^{p_0}\left(\log^+\sum_{j=1}^{\infty}\xi_j^{p_0}\right)\right)\pi(dx),$$

and it follows readily from the Malthusian hypotheses and the fact that $\sum_{|u|=n} \xi_u^{pp_0}$ is a supermartingale, that this quantity is finite. Therefore the assumption of Theorem 6.1 of [15] are check, which gives by Theorem 6.1 of [15] the second point and by Theorem 6.3 of [15] we get the third point. □

Acknowledgements: I wish to thank J. Bertoin for his help and suggestions. I also wish to thank the anonymous referees of an earlier draft for their detailed comments and suggestions.

References

[1] S. Asmussen and N. Kaplan (1976). Branching random walks. I. *Stochastic Process. Appl.* **4** , no. 1, 1-13.

[2] S. Asmussen and N. Kaplan (1976). Branching random walks. II. *Stochastic Process. Appl.* **4** , no. 1, 15-31.

[3] K. B. Athreya and P. E. Ney (1972). *Branching processes.* Springer-Verlag Berlin Heidelberg.

[4] J. Bertoin (2006). *Random fragmentation and coagulation processes.* Cambridge Univ. Pr.

[5] J. Bertoin (2006). Different aspects of a random fragmentation model. *Stochastic Process. Appl.* **116**, 345-369.

[6] J. Bertoin and A. V. Gnedin (2004) Asymptotic laws for nonconservative self-similar fragmentations. *Electron. J. Probab.* **9** , No. 19, 575-593

[7] J. Bertoin and M. Yor (2002). The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Analysis* **17** 389-400.

[8] J. D. Biggins (1992). Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20**, No. 1, 131-151.

[9] M. D. Brennan and R. Durrett (1986). Splitting intervals. *Ann. Probab.* **14** , No. 3, 1024-1036.

[10] M. D. Brennan and R. Durrett (1987). Splitting intervals. II. Limit laws for lengths. *Probab. Theory Related Fields* **75** No. 1, 109-127.

[11] M.E. Caballero and L. Chaumont (2006). Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.* **34**, No. 3, 1012-1034.

[12] B. Chauvin (1986). Arbres et processus de Bellman-Harris. *Ann. Inst. Henri Poincaré.* **22**, No. 2, 209-232.

[13] B. Chauvin (1991). Product martingales and stopping lines for branching Brownian motion. *Ann. Probab.* **19**, No. 3, 1195-1205.

[14] T. E. Harris (1963). *The theory of branching processes.* Springer

[15] P. Jagers (1989). General branching processes as Markov fields. *Stochastic Process. Appl.* **32**, 183-212.

[16] J. F. C. Kingman (1993). *Poisson processes.* Oxford Studies in Probability, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press.

[17] N. Krell (2008). Multifractal spectra and precise rates of decay in homogeneous fragmentations. *To appear in Stochastic Process. Appl.*

[18] A. E. Kyprianou (1999). A note on branching Lévy processes. *Stochastic Process. Appl.* **82** , No. 1, 1-14.

[19] A. E. Kyprianou (2000). Martingale convergence and the stopped branching random walk. *Probab. Theory Related Fields* **116** , no. 3, 405-419.

[20] O. Nerman (1984). *The growth and composition of supercritical branching populations on general type spaces.* Technical report, Dept. Mathematics, Chalmers Univ. Technology and Goteborg Univ.

[21] J. Neveu (1986). Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, No. 2, 199-207.